

# Topics on Hamiltonian Mechanics

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## Abstract

We modify Hamiltonian mechanics. We reformulate the law of conservation of energy.

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This paper is based on the following observation. Consider the following modified Hamiltonian equations

$$\frac{\partial Q}{\partial t_1} + \frac{\partial Q}{\partial t_2} = \frac{\partial H}{\partial P}, \quad \frac{\partial P}{\partial t_1} + \frac{\partial P}{\partial t_2} = -\frac{\partial H}{\partial Q} \quad (0.1)$$

for unknown functions  $Q = Q(t_1, t_2)$ ,  $P = P(t_1, t_2)$  and given Hamiltonian  $H = H(Q, P)$ .

If we set  $q(t) := Q(t, t)$ ,  $p(t) := P(t, t)$  then since  $\dot{q}(t) = \frac{\partial Q}{\partial t_1}(t, t) + \frac{\partial Q}{\partial t_2}(t, t)$  and  $\dot{p}(t) = \frac{\partial P}{\partial t_1}(t, t) + \frac{\partial P}{\partial t_2}(t, t)$  we conclude that  $q(t)$  and  $p(t)$  satisfy the classical Hamilton's equations. Thus if we computes the energy  $H(q(t), p(t))$  then we get a constant  $H(q(t), p(t)) = H(q(0), p(0))$ .

But instead of the above method, i.e. before putting  $t_1 = t_2 = t$  if we expand  $Q$  and  $P$  in terms of powers of  $t_2$  as

$$Q(t_1, t_2) = \sum_{n=0}^{\infty} q_n(t_1)t_2^n, \quad P(t_1, t_2) = \sum_{n=0}^{\infty} p_n(t_1)t_2^n, \quad (0.2)$$

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and then substitute them in the equations (0.1) and then by equating the coefficients of powers of  $t_2$  in two sides of each equation, we get some recursive relations among the unknown coefficients  $q_n(t_1), p_n(t_1)$  which can be solved by knowing the initial conditions  $q_0(t_1), p_0(t_1)$  and then substituting these coefficients in (0.2) we obtain  $Q = Q(t_1, t_2)$  and  $P = P(t_1, t_2)$  and at last by setting  $t_1 = t_2 = t$  we obtain  $q(t) := Q(t, t), p(t) := P(t, t)$ . In all examples which we were able to solve the equations by this method, we observed that after calculating the energy  $H(q(t), p(t))$  which as mentioned above that in general one expects to get a constant, in fact we got

$$H(q(t), p(t)) = \hat{h}(t), \quad (0.3)$$

where  $h(t) := H(q_0(t), p_0(t))$  and

$$\hat{h}(t) := \sum_{n=0}^{\infty} (-1)^n \frac{h^{(n)}(t)}{n!} t^n. \quad (0.4)$$

By  $h^{(n)}$  we mean the  $n$ -th derivative of  $h$ .

Since we expect constant energy, the question arises that if for any function  $f(t)$  one has  $\hat{f}(t) = f(0)$  for all  $t$ ? In fact if we differentiate the series term by term we get

$$\begin{aligned} \frac{d}{dt} \hat{f}(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(t)}{n!} t^n + \sum_{n=1}^{\infty} (-1)^n \frac{f^{(n)}(t)}{(n-1)!} t^{n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(t)}{n!} t^n - \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(t)}{n!} t^n \\ &= 0 \end{aligned}$$

Thus  $\hat{f}(t)$  should be constant. But the point is that we are not allowed to differentiate term by term from a series even the series is uniformly convergent.

We were not able to prove this up to the present time. The only result is

**Theorem 1** *If a function  $f$  can be expanded around origin via power series  $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n, t \in \mathbb{R}$ , then  $\hat{f}$  is absolutely and uniformly convergent to the constant  $f(0)$ . That is*

$$\sum_{n=0}^{\infty} (-1)^n \frac{f^{(n)}(t)}{n!} t^n = f(0), \quad t \in \mathbb{R}, \quad (0.5)$$

*for analytic functions.*

Another evidence for strangeness of the series  $\hat{f}(t)$  comes from the following argument. Suppose that  $f$  is a smooth periodic function which by the Fourier analysis we know that it has Fourier expansion. That is we suppose

$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{im\omega t}$ . Then it is well known that we can differentiate to get  $f^{(n)}(t) = \sum (im\omega)^n c_m e^{im\omega t}$ . Thus

$$\begin{aligned}\hat{f}(t) &= \sum_n \sum_m \frac{(-1)^n (im\omega)^n}{n!} c_m e^{im\omega t} t^n \\ &= \sum_m \sum_n \left( \frac{(-1)^n (im\omega)^n}{n!} t^n \right) c_m e^{im\omega t} \\ &= \sum_m e^{-im\omega t} c_m e^{im\omega t} \\ &= \sum_{m=-\infty}^{\infty} c_m \\ &= f(0).\end{aligned}$$

But the above argument is again analytically ill, since we have exchanged the order of two infinite sums which from the theorems of mathematical analysis we are not allowed in general to do so. In order to be able to do so there exists a general theorem on double series which states that if for a double infinite series  $\sum_{m,n} a_{mn}$  for each  $n$  the series  $\sum_m |a_{mn}|$  is convergent which we show its sum by  $b_n$  and the series  $\sum_n b_n$  is also convergent then we are allowed to exchange the order of summation. That is we have  $\sum_n \sum_m a_{mn} = \sum_m \sum_n a_{mn}$ . Now let us check if this criterion can be applied to the double series whose entries are  $a_{mn} := \frac{(-im)^n}{n!} c_m e^{imt} t^n$ . For simplicity we have assumed that  $\omega = 1$ . We have  $\sum_m |a_{mn}| = 2 \frac{t^n}{n!} \sum_{m=1}^{\infty} m^n |c_m|$ . But in Fourier analysis it is well known that the series  $f^{(n)}(t) = \sum (im)^n c_m e^{imt}$  is absolutely convergent. That is the series  $\sum_{m=1}^{\infty} m^n |c_m|$  is convergent which we show its sum by  $\alpha_n$ . Thus we have  $\sum_m |a_{mn}| = 2 \frac{\alpha_n t^n}{n!}$ . Thus we should verify the convergence of the series  $\sum_{n=0}^{\infty} \frac{\alpha_n t^n}{n!}$ . But this is a power series whose convergence radius is given by  $R = \alpha^{-1}$  where  $\alpha := \limsup \sqrt[n]{\frac{\alpha_n}{n!}} = \limsup \sqrt[n]{\frac{\alpha_n}{n!}}$ . The point is that we are not sure if  $R \neq 0$ ?

In summary the question of convergence of the series  $\hat{f}$  and as well as the constancy of the sum of the series for a periodic or a general smooth function is remain open.

**Open Questions** Is it true that for all systems (0.3) holds?

Is there non-analytic function  $f$  such that the series  $\hat{f}$  is point-wise or uniformly convergent and among such functions if there is any, is there function such that the series converges to a non-constant function?

We conjecture that there are nonanalytic probably periodic functions  $f$  such that the series  $\hat{f}(t)$  is piecewise constant and takes discrete values. That is  $\hat{f}(t)$  is constant only on some intervals and its values jump when one passes from an interval to its next interval. More details will appear soon in arXive.

Thus if  $q_0(t), p_0(t)$  are simultaneously analytic, that is if both can be expanded as power series

$$q_0(t) = \sum_{n=0}^{\infty} q_{0n} t^n, \quad p_0(t) = \sum_{n=0}^{\infty} p_{0n} t^n$$

then the function  $h(t) := H(q_0(t), p_0(t))$  is also analytic and therefore we recover the classical conservation law  $H(q(t), p(t)) = h(0) = H(q_0(0), p_0(0)) = H(q_{00}, p_{00})$ . Thus we observe that only the coefficients  $q_{00}$  and  $p_{00}$  of the expansion of  $q_0(t)$  and  $p_0(t)$  contribute in the energy. That is in the case in which  $q_0(t), p_0(t)$  are simultaneously analytic, dependence of  $q_0(t), p_0(t)$  to time does not effect the system and we can safely, as in the classical mechanics, assume that  $q_0(t) = q_0, p_0(t) = p_0$  are constants. That is from the beginning we did not need two dimensional time  $(t_1, t_2)$  and the modified equations (0.1) are superfluous and the classical Hamilton's equations are enough. But remember that we were able only prove the constancy of  $\hat{h}(t)$  only for simultaneously analytic primitive conditions  $q_0(t), p_0(t)$ . Thus if one of them is not analytic then we may find new feature in our model. So we need to study more such a strange object as (0.4)!

The following proposition will be useful in computations.

**Theorem 2** (i) If the series  $\hat{f}$  and  $\hat{g}$  are convergent then the series  $\widehat{af + bg}$  are also convergent and  $\widehat{af + bg} = a\hat{f} + b\hat{g}$   
(ii) If moreover at least one of series  $\hat{f}$  and  $\hat{g}$  are absolutely convergent then the series  $\widehat{fg}$  is also convergent and  $\widehat{fg} = \hat{f}\hat{g}$ .

## 1 Examples

**Example 3**  $H(q, p) = ap + bq$ , where  $a$  and  $b$  are some given constants.

**Proof** The recursive equations obtained from (0.1) after expansion (0.2) are  $(n+1)q_{n+1} + \dot{q}_n = a, (n+1)p_{n+1} + \dot{p}_n = -b$ . One can easily show by induction that  $q_n(t_1) = \frac{(-1)^n}{n!} q_0^{(n)}(t_1), p_n(t_1) = \frac{(-1)^n}{n!} p_0^{(n)}(t_1)$  for  $n \neq 1$  and  $q_1(t_1) = a - \dot{q}_0(t_1), p_1(t_1) = \dot{p}_0(t_1) - b$ . Thus  $q(t) = Q(t, t) = \sum_{n=0}^{\infty} q_n(t) t^n = at + \hat{q}_0(t)$  and  $p(t) = -bt + \hat{p}_0(t)$ . Finally  $H(q(t), p(t)) = -abt + a\hat{p}_0(t) + bat + b\hat{q}_0(t) = a\hat{p}_0 + b\hat{q}_0(t) = \hat{h}(t)$  where  $h(t) = ap_0(t) + bq_0(t) = H(q_0(t), p_0(t))$ . ■

**Example 4** The harmonic oscillator  $H(q, p) = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2$ .

**Proof** The recursive equations obtained from (0.1) after expansion (0.2) are

$$(n+1)q_{n+1} + \dot{q}_n = p_n, \quad (n+1)p_{n+1} + \dot{p}_n = -\omega^2 q_n. \quad (1.1)$$

We show by induction that for  $n > 0$

$$q_n = \frac{(-1)^n}{n!} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i C_{2i}^n \omega^{2i} q_0^{(n-2i)} - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i C_{2i+1}^n \omega^{2i} p_0^{(n-2i-1)} \right) \quad (1.2)$$

and

$$p_n = \frac{(-1)^n}{n!} \left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i C_{2i}^n \omega^{2i} p_0^{(n-2i)} + \omega^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i C_{2i+1}^n \omega^{2i} q_0^{(n-2i-1)} \right) \quad (1.3)$$

where  $C_i^j := \frac{j!}{i!(j-i)!}$ .

First step of induction  $n = 1$ . We put  $n = 0$  in (1.1) to get  $q_1 = -\dot{q}_0 + p_0$  and  $p_1 = -\dot{p}_0 - \omega^2 q_0$ . On the other hand if we put  $n = 1$  in the right hand sides of (??) we get  $q_1 = -(\dot{q}_0 - p_0)$  and  $p_1 = -(\dot{p}_0 + \omega^2 q_0)$ . Thus the first step of induction is verified. Let (1.1) is true for  $n$  we prove it for  $n+1$ . First suppose  $n = 2k+1$  is odd. We have

$$\begin{aligned}
-\frac{(n+1)!}{(-1)^{n+1}} q_{n+1} &= \frac{n!}{(-1)^n} (-\dot{q}_n + p_n) \\
&= -\sum_{i=0}^k (-1)^i C_{2i}^n \omega^{2i} q_0^{(n-2i+1)} + \sum_{i=0}^k (-1)^i C_{2i+1}^n \omega^{2i} p_0^{(n-2i)} \\
&\quad + \sum_{i=0}^k (-1)^i C_{2i}^n \omega^{2i} p_0^{(n-2i)} + \omega^2 \sum_{i=0}^k (-1)^i C_{2i+1}^n \omega^{2i} q_0^{(n-2i-1)} \\
&= -\sum_{i=0}^k (-1)^i C_{2i}^n \omega^{2i} q_0^{(n-2i+1)} + \omega^2 \sum_{i=1}^{k+1} (-1)^{i-1} C_{2(i-1)+1}^n \omega^{2(i-1)} q_0^{(n-2(i-1)-1)} \\
&\quad + \sum_{i=0}^k (-1)^i C_{2i+1}^n \omega^{2i} p_0^{(n-2i)} + \sum_{i=0}^k (-1)^i C_{2i}^n \omega^{2i} p_0^{(n-2i)} \\
&= -q_0^{(n+1)} + (-1)^k \omega^{2k+2} q_0 - \sum_{i=1}^k (-1)^i (C_{2i}^n + C_{2i-1}^n) \omega^{2i} q_0^{(n-2i+1)} \\
&\quad + \sum_{i=0}^k (-1)^i (C_{2i}^n + C_{2i+1}^n) \omega^{2i} p_0^{(n-2i)} \\
&= -q_0^{(n+1)} + (-1)^k \omega^{2k+2} q_0 - \sum_{i=1}^k (-1)^i C_{2i}^{n+1} \omega^{2i} q_0^{(n-2i+1)} \\
&\quad + \sum_{i=0}^k (-1)^i C_{2i+1}^{n+1} \omega^{2i} p_0^{(n-2i)} \\
&= -\sum_{i=0}^{k+1} (-1)^i C_{2i}^{n+1} \omega^{2i} q_0^{(n-2i+1)} + \sum_{i=0}^k (-1)^i C_{2i+1}^{n+1} \omega^{2i} p_0^{(n-2i)}
\end{aligned}$$

Thus  $q_{n+1} = \frac{(-1)^{n+1}}{(n+1)!} (\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i C_{2i}^{n+1} \omega^{2i} q_0^{(n-2i+1)} - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i C_{2i+1}^{n+1} \omega^{2i} p_0^{(n-2i)})$ .

This means that the part (1.2) of assertion of induction holds for  $n+1$  if assertion of induction holds for odd  $n$ . Similarly one can show that the part (1.3) of assertion of induction holds for  $n+1$  if assertion of induction holds for odd  $n$ . The case of even  $n$  is similar.

Moreover we show that if the primitive conditions  $q_0(t)$  and  $p_0(t)$  are such that the series  $\widehat{q}_0(t)$  and  $\widehat{p}_0(t)$  converges absolutely then

$$q(t) = \widehat{q}_0(t) \cos \omega t + \omega^{-1} \widehat{p}_0(t) \sin \omega t \quad (1.4)$$

and

$$p(t) = \widehat{p}_0(t) \cos \omega t - \omega \widehat{q}_0(t) \sin \omega t. \quad (1.5)$$

Since the series  $\cos(\omega t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \omega^{2n} t^{2n}$  and  $\sin(\omega t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \omega^{2n+1} t^{2n+1}$  are absolutely convergent, by the Mertens' theorem [1], we can multiply the following series by the Cauchy's product rule

$$\begin{aligned}
\widehat{q}_0(t) \cos \omega t + \omega^{-1} \widehat{p}_0(t) \sin \omega t &= \sum_{n=0}^{\infty} (-1)^n \frac{q_0^{(n)}(t)}{n!} t^n \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \omega^{2n} t^{2n} \\
&+ \omega^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{p_0^{(n)}(t)}{n!} t^n \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \omega^{2n+1} t^{2n+1} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left( \frac{(-1)^{n-2i}}{(n-2i)!} q_0^{(n-2i)} \frac{(-1)^i}{(2i)!} \omega^{2i} \right) t^{n-2i} t^{2i} \\
&+ \omega^{-1} \sum_{n=0}^{\infty} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left( \frac{(-1)^{n-2i-1}}{(n-2i-1)!} p_0^{(n-2i-1)} \frac{(-1)^i}{(2i+1)!} \omega^{2i+1} \right) t^{n-2i-1} t^{2i+1} \\
&= \sum_{n=0}^{\infty} \left( \frac{(-1)^n}{n!} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i C_{2i}^n \omega^{2i} q_0^{(n-2i)} \right. \\
&\quad \left. - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i C_{2i+1}^n \omega^{2i} p_0^{(n-2i-1)} \right) t^n \\
&= \sum_{n=0}^{\infty} q_n(t) t^n \\
&= Q(t, t) \\
&= q(t)
\end{aligned}$$

This proves (1.4). Similarly one can prove (1.5).

Thus  $H(q(t), p(t)) = \frac{1}{2} \widehat{p}_0^2 + \frac{\omega^2}{2} \widehat{q}_0^2 = \hat{h}(t)$ , where  $h(t) := \frac{1}{2} p_0^2(t) + \frac{\omega^2}{2} q_0^2(t) = H(q_0(t), p_0(t))$ . ■

**Example 5**  $H(q, p) = pq^2$ .

**Proof** The recursive equations obtained from (0.1) after expansion (0.2) are

$$(n+1)q_{n+1} + \dot{q}_n = \sum_{i=0}^n q_i q_{n-i}, \quad (n+1)p_{n+1} + \dot{p}_n = -2 \sum_{i=0}^n q_i p_{n-i}. \quad (1.6)$$

Solution of these equations are difficult. So instead of solving them, we just show that the few first terms of the series  $q(t) = Q(t, t) = \sum_{n=0}^{\infty} q_n(t) t^n$  and  $\frac{\hat{q}_0(t)}{1-t\hat{q}_0(t)}$  coincide and also the few first terms of the series  $p(t) = P(t, t) = \sum_{n=0}^{\infty} p_n(t) t^n$  and  $\hat{p}_0(t)(1-t\hat{q}_0(t))^2$  coincide. That is we conjecture that

$$q(t) = \frac{\hat{q}_0(t)}{1-t\hat{q}_0(t)}, \quad p(t) = \hat{p}_0(t)(1-t\hat{q}_0(t))^2. \quad (1.7)$$

By solving the first few terms of these recursive relations we get  $q_1 = q_0^2 - q_0'$ ,  $q_2 = q_0^3 - 2q_0q_0' + \frac{1}{2}q_0''$ ,  $q_3 = q_0^4 - 3q_0^2q_0' + q_0q_0'' + q_0'^2 - \frac{1}{6}q_0'''$ ,  $\dots$

On the other hand if we set  $\sum_{n=0}^{\infty} a_n t^n := \frac{\dot{q}_0(t)}{1-t\dot{q}_0(t)}$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} q_0^{(n)} t^n &= \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} q_0^{(n)} t^{n+1}\right) \sum_{n=0}^{\infty} a_n t^n \\ &= a_0 + (a_1 - a_0 q_0) t + (a_2 - a_1 q_0 + a_0 q_0') t^2 \\ &\quad + (a_3 - a_2 q_0 + a_1 q_0' - \frac{1}{2} a_0 q_0'') t^3 + \dots \end{aligned}$$

Thus  $a_0 = q_0$ ,  $a_1 - a_0 q_0 = -q_0'$ ,  $a_2 - a_1 q_0 + a_0 q_0' = \frac{1}{2} q_0''$ ,  $a_3 - a_2 q_0 + a_1 q_0' - \frac{1}{2} a_0 q_0'' = -\frac{1}{6} q_0'''$ ,  $\dots$ . Thus  $a_0 = q_0$ ,  $a_1 = q_0^2 - q_0'$ ,  $a_2 = q_0^3 - 2q_0q_0' + \frac{1}{2}q_0''$ ,  $a_3 = q_0^4 - 3q_0^2q_0' + q_0q_0'' + q_0'^2 - \frac{1}{6}q_0'''$ ,  $\dots$ . Hence we see that  $a_0 = q_0$ ,  $a_1 = q_1$ ,  $a_2 = q_2$ ,  $a_3 = q_3$ ,  $\dots$ . This proves the first relation of (1.7). The other part is proved similarly.

Up to the present time we have not been able to prove the conjecture (1.7). But suppose this conjecture is true. Then we have  $H(q(t), p(t)) = (\dot{q}_0(t))^2 \dot{p}_0(t) = \hat{h}(t)$  where  $h(t) = (q_0(t))^2 p_0(t) = H(q_0(t), p_0(t))$ . ■

## References

- [1] Tom M. Apostol *Mathematical Analysis*, Addison-Wesley, 1975